Applied GIS

ISSN: 1832-5505

Vol-10 Issue-02 July 2022

THE EQUIVALENCE OF TWO SEMI-FINITE FORMS OF THEQUINTUPLE PRODUCT IDENTITY

C CHAITANYA LAKSHMI, ASSISTANT PROFESSOR, chaithanyalakshmi216@gmail.com
D MADHUSUDANA REDDY, PROFESSOR, madhuskd@gmail.com
K SATYA NARAYANA, PROFESSOR, mastersatya08@gmail.com
Department of Mathematics, Sri Venkateswara Institute of Technology,
N.H 44, Hampapuram, Rapthadu, Anantapuramu, Andhra Pradesh 515722

Abstract

Through the application of Abel's summation by parts approach, it is demonstrated that two semi-finite forms of the quintuple product identity are comparable.

Keywords: quintuple product identity; semi-finite form; Abel’s method on summing by parts; q-series.

- **INTRODUCTION**

The celebrated quintuple product identity states that [1, p. 82]:

$$(1.1) \qquad \sum_{k=-\infty}^{\infty} (-1)^k q^{\frac{k(3k-1)}{2}} (1 + zq^k) z^{3k} = \frac{(q, q/z^2, z^2; q)_\infty}{(z, q/z; q)_\infty}, \qquad z \neq 0.$$

Here and throughout this note, we define the products of q-shifted factorials as usual by

$$(a; q)_\infty = \prod_{l=0}^{\infty} (1 - aq^l) \qquad \text{and} \qquad (a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty},$$

for $n \in \mathbb{Z}$ and $|q| < 1$, with the following abbreviated multiple parameter notation $(a, b, \cdots, c; q)_k = (a; q)_k (b; q)_k \cdots (c; q)_k$, $k \in \mathbb{Z} \cup \{\infty\}$.

For the historical remark and numerous proofs of this significant identity (1.1), the reader can consult the study [5]. Liu [6] provided a strong generalisation of (1.1) along with certain applications.

Two of the three semi-finite forms of the quintuple product identity provided by the author and Zhang [7, 8] are expressed in the ensuing two theorems, respectively.

Theorem 1. ([7, 8]) It has holds

$$(1.2) \qquad \sum_{k=0}^{\infty} \frac{(z^2; q)_k}{(q; q)_k} q^{k^2} (1 + zq^k) z^k = (-z; q)_\infty (z^2 q; q^2)_\infty,$$

ISSN: 1832-5505
Vol-10 Issue-02 July 2022

Theorem 1.2. ([8]) There holds

$$(1.3)\qquad \sum_{k=0}^{\infty} \frac{(z^2q;q)_k}{(q;q)_k} q^{k^2}(1 - z^2q^{2k+1})z^k = (-zq;q)_\infty (z^2q;q^2)_\infty, \qquad z \in \mathbb{C}.$$

We direct the reader to [7] and [8], respectively, for the technicalities of deriving the quintuple product identity (1.1) from Theorem 1.1 and 1.2.
Using Abel's summing by parts method, we shall demonstrate in this brief note that Theorem 1.2 is equal to Theorem 1.1.
When assessing finite and infinite summations, the modified version of Abel's lemma on summation by parts works incredibly well. See, for example, [2-4]. This approach is limited to the situation where the series is unilateral and nonterminating, as indicated by the lemma that follows.
For an arbitrary complex sequence $\{A_k\}$, let

$$\sum_{k=0}^{\infty} B_k \nabla A_k = [AB]_+ - A_{-1}B_0 + \sum_{k=0}^{\infty} A_k \triangle B_k,$$

$\nabla A_k := A_k - A_{k-1}$ and $\triangle A_k := A_k$

$- A_{k+1}$. Lemma 1.3. Let $\{A_k\}$ and $\{B_k\}$ be two complex sequences.

Then we have

provided that the series on both sides are convergent and there exists

the limit $[AB]_+ := \lim_{k\to\infty} A_k B_{k+1}$.

- **THE EQUIVALENCE of THEOREM 1.1 AND 1.2**

Let

$$f(z) := \sum_{k=0}^{\infty} \frac{(z^2q;q)_k}{(q;q)_k} q^{k^2}(1 - z^2q^{2k+1})z^k$$

And

$$g(z) := \sum_{k=0}^{\infty} \frac{(z^2;q)_k}{(q;q)_k} q^{k^2}(1 + zq^k)z^k.$$

ISSN: 1832-5505
Vol-10 Issue-02 July 2022

Define

$$A_k := \frac{(q^2z^2;q)_k}{(q;q)_k}(-1)^k q^{\frac{k^2+k}{2}} \qquad \text{and} \qquad B_k := (-1)^k q^{\frac{k^2+k}{2}} z^k.$$

Then, we have

With the differences

And

Using Lemma 1.3, we get

$$A_{-1}B_0 = [AB]_+ = 0$$

$$\nabla A_k = \frac{(1-z^2q^{2k+1})(z^2q;q)_k}{(1-z^2q)(q;q)_k}(-1)^k q^{\frac{k^2-k}{2}}$$

$$\vartriangle B_k = (-1)^k q^{\frac{k^2+k}{2}}(1+zq^{k+1})z^k.$$

$$f(z) = (1-z^2q)\sum_{k=0}^{\infty} B_k \nabla A_k = (1-z^2q)\sum_{k=0}^{\infty} A_k \vartriangle B_k$$

$$= (1 - z^2 q) \sum_{k=0}^{\infty} \frac{(q^2 z^2; q)_k}{(q; q)_k} q^{k^2+k} (1 + zq^{k+1}) z^k$$

$$= (1 - z^2 q) g(qz). \tag{2.1}$$

- From Theorem 1.2 to Theorem 1.1. Combining (2.1) and Theorem 1.2, we get

$$g(qz) = \frac{f(z)}{1 - z^2 q} = \frac{(-zq; q)_\infty (z^2 q; q^2)_\infty}{1 - z^2 q} = (-zq; q)_\infty (z^2 q^3; q^2)_\infty.$$

Then, replacing z with zq−1 gives the identity (1.2), which completes the proof of Theorem 1.1.

- From Theorem 1.1 to Theorem 1.2. Using (2.1) and Theorem 1.1, we have

$f(z) = (1 - z^2 q) g(qz) = (1 - z^2 q)(-zq; q)_\infty (z^2 q^3; q^2)_\infty = (-zq; q)_\infty (z^2 q; q^2)_\infty$, which is the identity(1.3). This ends the proof of Theorem 1.2.

Acknowledgment. This work is partially supported by the Major Project of Henan's Universitites (Grant No.23A110013) and the National Natural Science Foundation of China (Grant No. 11871258).

References

- – B.C. Berndt, Part III of Ramanujan's Notebook, Springer-Verlag, New York, 1991.
- Abels technique on summing by parts and terminating well-poised q- series identities, W. Chu, C.Z. Jia, 360–370; J. Comput. Appl. Math. 207 (2007).
- The Abels technique on summing by parts and theta hypergeometric series by W. Chu and C.Z. Jia Theory Ser. A 115 (2008) 815–844. J. Comb.
- • W. Chu, X. Wang, Abels lemma on terminating q-series identities and summation by parts, Numer Algor 49 (2008) 105–128.
- S. Cooper, "The identity of the quintuple product," International Journal of Number Theory, 2 (1) (2006), 115–161.
- An expansion of the quintuple product identification and its uses is presented by Z.-G. Liu. Pacific (2010) J. Math. 246 (2), 345–390
- A semi-finite form of the quintuple product identity, J.-M. Zhu, Z.-Z. Zhang https://doi.org/10.1016/j.jcta.2021.105509, J. Comb. Theory Ser. A 184 (2021) 105509.
- A comment on semi-finite forms of the quintuple product identity (J.-M. Zhu, Z.-Z. Zhang), 5.1016/j.jmaa.2022.126350, J. Math. Anal. Appl. 514 (2022) 126350 |.